

On the maximum nilpotent orbit intersecting a centralizer in $M(n, K)$

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Abstract

To any pair of commuting $n \times n$ nilpotent matrices it is associated a pair of partitions of n . Recently several authors have published results about the problem of finding which pairs of partitions correspond to pairs of commuting nilpotent matrices. We describe a maximal nilpotent subalgebra of the centralizer of a given nilpotent $n \times n$ matrix. Then, using a property of the orbits which intersect some simple types of nilpotent subalgebras, we prove that the maximum partition which forms with a given partition a pair with the previous property can be found by a simple algorithm which was conjectured by Polona Oblak.

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1 Introduction

We will use the following notations: $M(n, K)$ is the set of the $n \times n$ matrices over a field K , $\text{GL}(n, K)$ is the set of the $n \times n$ nonsingular matrices over K , $N(n, K)$ is the set of the $n \times n$ nilpotent matrices over K , $J \in N(n, K)$ is a matrix with Jordan canonical form, $\mu_1 \geq \mu_2 \geq \dots \geq \mu_t$ are the orders of the Jordan blocks of J , $B = (\mu_1, \dots, \mu_t)$ is the partition of n associated to J and the orbit of J under the action of $\text{GL}(n, K)$.

Let J' be another nilpotent matrix and let $\mu'_1 \geq \dots \geq \mu'_{t'}$ be the orders of its Jordan block. We set $B' = (\mu'_1, \dots, \mu'_{t'})$. Then $B = B'$ iff $\text{rank } J^m = \text{rank } (J')^m$ for all $m \in \mathbb{N}$.

It is said that $B < B'$ if $\text{rank } J^m \leq \text{rank } (J')^m$ for all $m \in \mathbb{N}$ and there exists $m \in \mathbb{N}$ such that $\text{rank } J^m < \text{rank } (J')^m$.

The following claim is due to Hesselink ([7], 1976): $B < B'$ iff B , as an orbit, is contained in the closure of the orbit B' .

For $i \in \mathbb{N}$ we set $\mu_i = 0$ for $i > t$, $\mu'_i = 0$ for $i > t'$; then we have that

$$B \leq B' \iff \sum_{i=1}^l \mu_i \leq \sum_{i=1}^l \mu'_i \text{ for all } l \in \mathbb{N}$$

where equal holds in the first relation iff it holds in the second one for all $l \in \mathbb{N}$.

Examples $(6, 4, 3) < (6, 5, 2) < (6, 6, 1)$, $(5, 3, 2, 1) < (6, 3, 1, 1) < (6, 4, 1)$.

Let \mathcal{C}_B be the centralizer of J and let \mathcal{N}_B be the subset of \mathcal{C}_B of all the nilpotent matrices. We recall the following result, whose proof is a consequence of Wedderburn's theorems.

Lemma 1.1 *If \mathcal{U} is a finite dimensional algebra over an infinite field K then the scheme $\mathcal{N}(\mathcal{U})$ of nilpotent elements of \mathcal{U} is an irreducible variety.*

For $m \in \mathbb{N}$ the subvariety of \mathcal{N}_B of all X such that $\text{rank } X^m$ is the maximum possible is open and, by lemma 1.1, \mathcal{N}_B is irreducible, then the intersection of these open subsets for $m \in \mathbb{N}$ is non-empty. Hence there is a maximum partition for the elements of \mathcal{N}_B and the subset of the elements which have this partition is open (dense) in \mathcal{N}_B . Then it is defined a map Q in the set of the orbits of $n \times n$ nilpotent matrices (or partitions of n) which associates to any orbit B the maximum nilpotent orbit which intersects \mathcal{N}_B .

Let R be the $n \times n$ Jordan block. For $s \in \mathbb{N} - \{0\}$ let q and r be the quotient and the remainder of the division of n by s . We have that R^s has r Jordan blocks of order $q + 1$ and $s - r$ Jordan blocks of order q . Then the partition B is almost rectangular (that is $\mu_1 - \mu_t \leq 1$) iff J is conjugated to a power of R . Hence if B is almost rectangular we have $Q(B) = (n)$. As a consequence of the next proposition we have that the converse of this claim is also true.

Let $\{n_1, \dots, n_{r_B}\}$ be the ordered subset of $\{1, \dots, t\}$ such that $n_{r_B} = t$, $\mu_1 - \mu_{n_1} \leq 1$ and $\mu_{n_{i-1}+1} - \mu_{n_i} \leq 1$, $\mu_{n_{i-1}} - \mu_{n_i} > 1$ for $i = 2, \dots, r_B$. Then r_B is the minimum possible $p \in \mathbb{N}$ such that there exist almost rectangular partitions B_1, \dots, B_p such that $B = (B_1, \dots, B_p)$.

Examples If $B = (5, 4, 3, 1, 1)$ we have $r_B = 3$, $n_1 = 1$, $n_2 = 3$; if $B = (9, 7, 5, 1)$ we have $r_B = 4$.

Proposition 1.1 *There exists a non-empty open subset of \mathcal{N}_B such that if A belongs to it we have that $\text{rank } A = n - r_B$ (that is A has r_B Jordan blocks).*

Proof See [4] (2003). \square

Let s_B be the maximum of the cardinalities of the almost rectangular subpartitions of B .

Lemma 1.2 For $A \in \mathcal{N}_B$ and $m \in \mathbb{N}$ we have

$$\text{rank } (A^{s_B})^m \leq \text{rank } J^m .$$

Proof See [4] (2003). \square

Let $\tilde{\mu}_1 = \mu_1 + \dots + \mu_{n_1}$ and $\tilde{\mu}_i = \mu_{n_{i-1}+1} + \dots + \mu_{n_i}$ for $i = 2, \dots, r_B$; let $\tilde{B} = (\tilde{\mu}_1, \dots, \tilde{\mu}_{r_B})$. By lemma 1.2 we get the following result (see [5]).

Proposition 1.2 If $s_B = n_i - n_{i-1}$ for $i = 1, \dots, r_B$ then $Q(B) = \tilde{B}$.

Example If $B = (5, 4, 4, 2, 2, 1)$ we have $\tilde{B} = (13, 5)$ and $Q(B) = \tilde{B}$.

Corollary 1.1 We have that $Q(B) = B$ iff $r_B = t$, that is $\mu_i - \mu_{i+1} > 1$ for $i = 1, \dots, t-1$.

The map Q was investigated by D.I. Panyushev in [13], in the more general context of Lie algebras, with the following result. Let \mathfrak{g} be a semisimple Lie algebra over an algebraically closed field K such that $\text{char } K = 0$; let G be its adjoint group and let $\mathcal{N}(\mathfrak{g})$ be the nilpotent cone of \mathfrak{g} . B. Kostant in [9] (1963) proved that $\mathcal{N}(\mathfrak{g})$ is irreducible. Let $e \in \mathcal{N}(\mathfrak{g})$ and let $\mathfrak{z}_{\mathfrak{g}}(e)$ be the centralizer of e . The element e is said "self-large" if $G \cdot e \cap (\mathfrak{z}_{\mathfrak{g}}(e) \cap \mathcal{N}(\mathfrak{g}))$ is open (dense) in $\mathfrak{z}_{\mathfrak{g}}(e) \cap \mathcal{N}(\mathfrak{g})$. Let $\{e, h, f\}$ be an \mathfrak{sl}_2 -triple and let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ be the corresponding \mathbb{Z} -grading of \mathfrak{g} . D. I. Panyushev in [13] (2008) proved the following result.

Theorem 1.1 The element $e \in \mathcal{N}$ is "self-large" iff $\mathfrak{z}_{\mathfrak{g}}(e) \cap \mathfrak{g}(0)$ is toral and $\mathfrak{z}_{\mathfrak{g}}(e) \cap \mathfrak{g}(1) = \{0\}$.

Let $\mathcal{H}(\mathfrak{g}) = \{(x, y) \in \mathcal{N}(\mathfrak{g}) \times \mathcal{N}(\mathfrak{g}) \mid [x, y] = 0\}$. The motivation and the proof of the previous theorem are linked to some results included a paper of A. Premet ([14], 2003), where it is proved the following result.

Theorem 1.2 The variety $\mathcal{H}(\mathfrak{g})$ ($\text{char } K \geq 0$) is equidimensional; there is a bijection between the set of the distinguished nilpotent orbits of \mathfrak{g} and the set of the irreducible components of $\mathcal{H}(\mathfrak{g})$.

The characterization of "self-large" orbits in $N(n, K)$ (corollary 1.1) was used in the proof of the following result, due to Tomaž Košir and Polona Oblak ([10], 2008).

Theorem 1.3 *For any partition B the partition $Q(B)$ has decreasing parts differing by at least 2, hence the map Q is idempotent.*

Let $B = (\mu_1, \dots, \mu_t)$ and for $i = 1, \dots, t$ let s_i be the maximum element of $\{i, \dots, t\}$ such that $\mu_i - \mu_{s_i} \leq 1$. Polona Oblak ([12], 2007) proved the following result.

Theorem 1.4 *The maximum index of nilpotency for an element of \mathcal{N}_B (that is the first number of the partition $Q(B)$) is the following:*

$$\nu_1 = \max_{i=1, \dots, t} \{2(i-1) + \mu_i + \mu_{i+1} + \dots + \mu_{s_i}\}.$$

Example If $B = (5^2, 4, 3^4, 2, 1)$ the maximum which appears in the previous equality is obtained for $i = 3$ and we have that

$$\nu_1 = 2 \times 2 + 4 + 3 \times 4 = 20.$$

Let $\{q_1, q_2, \dots, q_u\}$ be the ordered subset of $\{0, \dots, t\}$ such that $q_u = t$ and

$$\mu_1 = \mu_{q_1} \neq \mu_{q_1+1} = \mu_{q_2} \neq \mu_{q_2+1} = \dots \neq \mu_{q_{u-1}+1} = \mu_{q_u}$$

(for example if $B = (6, 6, 6, 6, 5, 2, 2, 1)$ we have that $q_1 = 4, q_2 = 5, q_3 = 7, q_4 = 8$). If we set $q_0 = 0$ this means that J has $q_i - q_{i-1}$ Jordan blocks of order μ_{q_i} for $i = 1, \dots, u$. We will write the partition (μ_1, \dots, μ_t) also as

$$(\mu_{q_1}^{q_1}, \mu_{q_2}^{q_2 - q_1}, \dots, \mu_{q_u}^{q_u - q_{u-1}}).$$

We will write the canonical basis Δ_B of K^n in the following way:

$$\Delta_B = \{v_{\mu_{q_i}, j}^{\mu_{q_i}}, v_{\mu_{q_i}, j}^{\mu_{q_i} - 1}, \dots, v_{\mu_{q_i}, j}^1, i = 1, \dots, u, j = q_i - q_{i-1}, \dots, 1\}.$$

For example if $B = (5, 3, 3, 2, 1) = (5^1, 3^2, 2^1, 1^1)$ we will write

$$\Delta_B = \{ \underbrace{v_{5,1}^5, v_{5,1}^4, \dots, v_{5,1}^1}_{\text{5 blocks}}, \underbrace{v_{3,2}^3, v_{3,2}^2, v_{3,2}^1}_{\text{3 blocks}}, \underbrace{v_{3,1}^3, v_{3,1}^2, v_{3,1}^1}_{\text{3 blocks}}, \underbrace{v_{2,1}^2, v_{2,1}^1}_{\text{2 blocks}}, \underbrace{v_{1,1}^1}_{\text{1 block}} \}.$$

Let $\tilde{i} \in \{1, \dots, u\}$ and $s \in \{0, 1\}$ be such that:

$$\tilde{a}) \quad \tilde{i} + s \in \{1, \dots, u\}, \mu_{q_{\tilde{i}}} - \mu_{q_{\tilde{i}+s}} \leq 1;$$

$\tilde{b})$ the set Δ_B° which is the union of the subsets:

$$\Delta_B^{\circ,1} = \{v_{\mu_{q_i}, j}^1, v_{\mu_{q_i}, j}^{\mu_{q_i}} \mid j = q_i - q_{i-1}, \dots, 1, i = 1, \dots, \tilde{i} - 1\},$$

$$\Delta_B^{\circ,2} = \{v_{\mu_{q_i}, j}^l \mid j = q_i - q_{i-1}, \dots, 1, l = 1, \dots, \mu_{q_i}, i = \tilde{i}, \tilde{i} + s\}$$

has the maximum possible cardinality.

Let \widehat{B} be the partition obtained from B by cancelling the powers $\mu_{q_i}^{q_i - q_{i-1}}$ for $i = \tilde{i}, \tilde{i} + s$ and by decreasing by 2 the numbers μ_{q_i} for $i = 1, \dots, \tilde{i} - 1$, that is:

$$\widehat{B} = ((\mu_{q_1} - 2)^{q_1}, \dots, (\mu_{q_{\tilde{i}-1}} - 2)^{q_{\tilde{i}-1} - q_{\tilde{i}-2}}, \mu_{q_{\tilde{i}+s+1}}^{q_{\tilde{i}+s+1} - q_{\tilde{i}+s}}, \dots, \mu_{q_u}^{q_u - q_{u-1}}) .$$

Let $Q(B) = (\nu_1, \dots, \nu_z)$; let $Q(\widehat{B}) = (\widehat{\nu}_1, \dots, \widehat{\nu}_{\widehat{z}})$. The main aim of this paper is to prove the conjecture of Polona Oblak which is expressed by the following theorem.

Theorem 1.5 *The maximum partition which is associated to elements of \mathcal{N}_B is*

$$(\nu_1, \nu_2, \dots, \nu_z) = (\nu_1, \widehat{\nu}_1, \widehat{\nu}_2, \dots, \widehat{\nu}_{\widehat{z}}) .$$

This conjecture was communicated by its author to some of the participants of the meeting "Fifth Linear Algebra Workshop" (Kranjska Gora, May 27 - June 5, 2008). A partial result on it was obtained with another approach in [8].

2 A property of the orbits which intersect some nilpotent subalgebras

We will study maps Φ in $\{1, \dots, n+1\}$ with the following property:

$$P) \quad \Phi(i) > i \text{ for } i = 1, \dots, n ; \quad \Phi(n+1) = n+1 .$$

Any map Φ can be associated to the subset N_Φ of $N(n, K)$ of all $X = (x_{i,j})$ such that $x_{i,l} = 0$ if $l < \Phi(i)$.

Lemma 2.1 *If Φ, Φ' are maps in $\{1, \dots, n+1\}$ with property P) then*

- a) $\Phi \circ \Phi'$ has property P);
- b) the image of $p : N_\Phi \times N_{\Phi'} \longrightarrow N(n, K)$ defined by $p(X, Y) = XY$ is $N_{\Phi \circ \Phi'}$;
- c) N_Φ is an algebra;
- d) N_Φ is stable with respect to the action of the group of all nonsingular upper triangular matrices.

Proof a) is obvious by the definition of P); b) can be deduced by considering the transpose matrices of X, Y ; c) is a consequence of b); d) can be proved as b), since the algebra of all upper triangular matrices corresponds to the map Ψ in $\{1, \dots, n+1\}$ defined by $\Psi(l) = l$ for $l = 1, \dots, n+1$. \square

We will also consider maps ϕ in $\{1, \dots, n+1\}$ with property P) and the following property:

Q) the restriction of ϕ to $\phi^{-1}(\{2, \dots, n\})$ is injective .

To each map ϕ with properties P) and Q) we associate the open subset \mathcal{A}_ϕ of N_ϕ defined by $x_{i, \phi(i)} \neq 0$ for all $i \in \phi^{-1}(\{2, \dots, n\})$.

Lemma 2.2 *If ϕ, ϕ' are maps in $\{1, \dots, n+1\}$ with properties P) and Q) then:*

- a) $\phi \circ \phi'$ has properties P) and Q);
- b) the maximum nilpotent orbit which has nonempty intersection with N_ϕ contains \mathcal{A}_ϕ .

Proof If $i, j \in (\phi \circ \phi')^{-1}(\{2, \dots, n\})$ and $\phi \circ \phi'(i) = \phi \circ \phi'(j)$ then $\phi'(i) = \phi'(j) \neq n+1$, hence $i = j$; this proves a). If $X \in \mathcal{A}_\phi$ the submatrix of X obtained by choosing, for all $i \in \phi^{-1}(\{2, \dots, n\})$, the row of index i and the column of index $\phi(i)$ has the maximum possible rank. Moreover for all $n \in \mathbb{N}$ we have that $X^n \in \mathcal{A}_{\phi^n}$, hence we get b). \square

In the set of all the maps in $\{1, \dots, n+1\}$ with property P) we can define a total order as follows:

$\Phi' < \Phi$ if there exists $i \in \{0, \dots, n\}$ such that

$$\Phi'(l) = \Phi(l) \text{ for all } l > i, \Phi(i) < \Phi'(i) .$$

To any map Φ in $\{1, \dots, n+1\}$ with property P) we can associate the set Σ_Φ of all the maps ϕ in $\{1, \dots, n+1\}$ with properties P) and Q) such that $\phi(l) \geq \Phi(l)$ for $l = 1, \dots, n+1$.

Lemma 2.3 *If Φ is a map in $\{1, \dots, n+1\}$ with property P) there exists a map $\phi \mapsto \mathcal{V}_{\Phi, \phi}$ from Σ_Φ to the set of the subsets of N_Φ such that:*

- a) if $X = (x_{i,j}) \in N_\Phi$ then $X \in \mathcal{V}_{\Phi, \phi}$ iff there exists $X' = (x'_{i,j}) \in \mathcal{A}_\phi$ which is conjugated to X by a nonsingular upper triangular matrix; moreover $x'_{i,j} = x_{i,j} + F_{i,j}$ where $F_{i,j}$ is a rational function of the entries $x_{h,k}$ such that either $h > i$ or $h = i$ and $k < i$, for $i, j \in \{1, \dots, n\}$;

$$b) \bigcup_{\phi \in \Sigma_\Phi} \mathcal{V}_{\Phi, \phi} = N_\Phi;$$

c) if ϕ is the maximum element of Σ_Φ then $\mathcal{V}_{\Phi, \phi}$ is a nonempty open subset of N_Φ ; otherwise $\mathcal{V}_{\Phi, \phi}$ is a nonempty open subset of

$$\bigcap_{\phi' \in \Sigma_\Phi, \phi' > \phi} (N_\Phi - \mathcal{V}_{\Phi, \phi'}).$$

Proof We can prove the claim by induction on n , hence we can assume that the claim is true for the restriction of Φ to $\{2, \dots, n+1\}$, which we will denote by $\Phi|$. Let $\{e_1, \dots, e_n\}$ be the canonical basis of K^n ; let $X = (x_{i,j}) \in N_\Phi$ and let $X|$ be the submatrix of X obtained by choosing the last $n-1$ rows and columns. By the inductive hypothesis there exists $\phi| \in \Sigma_{\Phi|}$ such that $X| \in \mathcal{V}_{\Phi|, \phi|}$. We can change the basis $\{e_2, \dots, e_n\}$ according to the inductive hypothesis a); hence we can assume that $X| \in \mathcal{A}_{\phi|}$. Let $(\Sigma_\Phi)|$ be the subset of Σ_Φ of all the elements whose restriction to $\{2, \dots, n+1\}$ is $\phi|$. For $l \in \phi|^{-1}(\{3, \dots, n\})$ we replace e_l with

$$e'_l = e_l + \frac{x_{1, \phi|(l)}}{x_{l, \phi|(l)}} e_1,$$

getting a new basis of K^n . Let $X' = (x'_{i,j})$ be the representation of X with respect to this new basis; then $x'_{1,l} = 0$ for all $l \in \phi|(\{2, \dots, n\}) - \{n+1\}$. Let $\phi \in (\Sigma_\Phi)|$ be defined as follows: $\phi(1)$ is the minimum of the set of all $l \in \{2, \dots, n+1\}$ such that $x'_{1,l} \neq 0$. We have that ϕ is the maximum element of $(\Sigma_\Phi)|$ if and only if $\phi(1)$ is the minimum of

$$\{\Phi(1), \dots, n+1\} - (\phi|(\{2, \dots, n\}) - \{n+1\});$$

if $\phi', \phi \in (\Sigma_\Phi)|$ then $\phi < \phi'$ if and only if $\phi(1) > \phi'(1)$. This, together with the inductive hypothesis c) on $\Phi|$, proves c). By d) of lemma 2.1 we get that if $X \in N_\Phi$ and X is conjugated to an element of \mathcal{A}_ϕ by a nonsingular upper triangular matrix then $X \in \mathcal{V}_{\Phi, \phi}$, for all $\phi \in \Sigma_\Phi$. \square

Corollary 2.1 *If \mathcal{W} is a nonempty subvariety of N_Φ there exists $\tilde{\phi} \in \Sigma_\Phi$ such that the set $\widetilde{\mathcal{W}}$ of all the elements of \mathcal{W} which are conjugated to elements of $\mathcal{A}_{\tilde{\phi}}$ by a nonsingular upper triangular matrix is a nonempty open subset of \mathcal{W} .*

Proof For $\phi \in \Sigma_\Phi$ let $\mathcal{V}_{\Phi, \phi}$ be as in lemma 2.3. Let $\Sigma_\Phi^\mathcal{W}$ be the subset of Σ_Φ of all ϕ such that $\mathcal{W} \cap \mathcal{V}_{\Phi, \phi} \neq \emptyset$, which by b) of lemma 2.3 isn't empty,

and let $\tilde{\phi}$ be the maximum element of $\Sigma_{\Phi}^{\mathcal{W}}$. Then we have that

$$\mathcal{W} \subset \bigcap_{\phi' \in \Sigma_{\Phi}, \phi' > \tilde{\phi}} \left(N_{\Phi} - \mathcal{V}_{\Phi, \phi'} \right)$$

and $\mathcal{V}_{\Phi, \tilde{\phi}}$ is a nonempty open subset of this intersection, which proves the claim. \square

3 The subalgebras \mathcal{E}_B , \mathcal{N}_B , $\overline{\mathcal{E}}_B$ and $\overline{\mathcal{N}}_B$

We will consider any $n \times n$ matrix X as a block matrix $(X_{h,k})$, where $X_{h,k}$ is a $\mu_h \times \mu_k$ matrix and $h, k = 1, \dots, t$, that is

$$X = \begin{pmatrix} X_{1,1} & X_{1,2} & \cdots & X_{1,t} \\ X_{2,1} & X_{2,2} & \cdots & X_{2,t} \\ \cdots & \cdots & \cdots & \cdots \\ X_{t,1} & X_{t,2} & \cdots & X_{t,t} \end{pmatrix}.$$

Let \mathcal{D}_B be the subalgebra of $M(n, K)$ of all X such that for $1 \leq k \leq h \leq t$ the blocks $X_{h,k}$ and $X_{k,h}$ have the following form:

$$X_{h,k} = \begin{pmatrix} 0 & \cdots & 0 & x_{h,k}^{1,1} & x_{h,k}^{2,1} & \cdots & x_{h,k}^{\mu_h,1} \\ \vdots & & & 0 & x_{h,k}^{1,2} & \ddots & x_{h,k}^{\mu_h-1,2} \\ \vdots & & & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & x_{h,k}^{1,\mu_h} \end{pmatrix},$$

$$X_{k,h} = \begin{pmatrix} x_{k,h}^{1,1} & x_{k,h}^{2,1} & \cdots & x_{k,h}^{\mu_h,1} \\ 0 & x_{k,h}^{1,2} & \ddots & x_{k,h}^{\mu_h-1,2} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & 0 & x_{k,h}^{1,\mu_h} \\ \vdots & & & 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

where for $\mu_h = \mu_k$ we omit the first $\mu_k - \mu_h$ columns and the last $\mu_k - \mu_h$ rows respectively.

For $X \in \mathcal{D}_B$, $i, j \in \{1, \dots, u\}$ and $l \in \{1, \dots, \mu_{q_1}\}$ let

$$X(i, j, l) = (x_{h,k}^{1,l}), \quad q_{i-1} + 1 \leq h \leq q_i, \quad q_{j-1} + 1 \leq k \leq q_j.$$

We set $X(i, l) = X(i, i, l)$. In the case of \mathcal{D}_B we have the following result, more precise than lemma 1.1.

Lemma 3.1 *For $X \in \mathcal{D}_B$ we have that:*

- a) *there exists $G \in GL(n, K)$ such that $G^{-1}XG(i, l)$ is lower (upper) triangular for $i = 1, \dots, u$ and $l = 1, \dots, \mu_{q_i}$;*
- b) *X is nilpotent if and only if $X(i, l)$ is nilpotent for $i = 1, \dots, u$ and $l = 1, \dots, \mu_{q_i}$.*

Proof It can be used a semisimple subalgebra of \mathcal{D}_B whose direct sum with the Jacobson radical of \mathcal{D}_B is \mathcal{D}_B ; the construction is as follows. For $l = 1, \dots, \mu_{q_1}$ let

$$U^l = \langle v_{\mu_{q_i}, j}^l : i = 1, \dots, u, j = q_i - q_{i-1}, \dots, 1, \mu_{q_i} \geq l \rangle;$$

then $K^n = \bigoplus_{l=1}^{\mu_{q_1}} U^l$ and $X(U^l) \subseteq \bigoplus_{i=l}^{\mu_{q_1}} U^i$. For $v \in K^n$ let $v = \sum_{l=1}^{\mu_{q_1}} v^{(l)}$ where $v^{(l)} \in U^l$ and let $L_{X,l} : U^l \rightarrow U^l$ be defined by $L_{X,l}(v) = X(v)^{(l)}$. Then X is nilpotent if and only if $L_{X,l}$ is nilpotent for $l = 1, \dots, \mu_{q_1}$. For $l = 1, \dots, \mu_{q_1}$ let $i_l \in \{1, \dots, u\}$ be such that $l \leq \mu_{q_{i_l}}$ and $\mu_{q_{i_l+1}} < l$ if $i_l \neq u$. Then the matrix of $L_{X,l}$ with respect to the basis $\{v_{\mu_{q_i}, j}^l : i = 1, \dots, u, \mu_{q_i} \geq l\}$ is the lower triangular block matrix $(X(i, j, l))$, $i, j = 1, \dots, i_l$, which is nilpotent if and only if $X(i, l)$ is nilpotent for $i = 1, \dots, i_l$. For $v \in K^n$ let $v = \sum_{i=1}^u v_{(i)}$ where $v_{(i)} \in \langle v_{\mu_{q_i}, j}^l : j = q_i - q_{i-1}, \dots, 1, l = 1, \dots, \mu_{q_i} \rangle$. For $i = 1, \dots, u$ and $l = 1, \dots, \mu_{q_i}$ let $U_i^l = \langle v_{\mu_{q_i}, j}^l : j = q_i - q_{i-1}, \dots, 1 \rangle$ and let $L_{X,i,l} : U_i^l \rightarrow U_i^l$ be defined by $L_{X,i,l}(v) = L_{X,l}(v)_{(i)}$. Then $X(i, l)$ is the matrix of $L_{X,i,l}$ with respect to the basis $\{v_{\mu_{q_i}, j}^l : j = q_i - q_{i-1}, \dots, 1\}$. We can substitute this basis with another basis of the same subspace such that $X(i, l)$ is upper triangular, for $i = 1, \dots, u$ and $j = q_i - q_{i-1}, \dots, 1$. \square

We will denote by $\overline{\mathcal{D}}_B$ the subspace of \mathcal{D}_B of all X such that $X(i, l)$ is lower triangular for $i = 1, \dots, u$ and $l = 1, \dots, \mu_{q_i}$. Moreover we will denote by \mathcal{E}_B the subset of \mathcal{D}_B of all the nilpotent matrices and we will set

$$\overline{\mathcal{E}}_B = \overline{\mathcal{D}}_B \cap \mathcal{E}_B.$$

Lemma 3.2 *We have that*

i) $X \in \mathcal{C}_B$ iff $X \in \mathcal{D}_B$ and

$$x_{h,k}^{l,1} = x_{h,k}^{l,2} = \dots = x_{h,k}^{l,\mu_h+1-l}, \quad x_{k,h}^{l,1} = x_{k,h}^{l,2} = \dots = x_{k,h}^{l,\mu_h+1-l}$$

for $1 \leq k \leq h \leq t$ and $l = 1, \dots, \mu_h$;

ii) if $X \in \mathcal{C}_B$ we can choose G with the property expressed in a) of lemma 3.1 and such that $GJ = JG$.

Proof For i) see [1] or [3]. Using the notations of the proof of lemma 3.1, for $i = 1, \dots, u$ let $(c_{h,k}^{(i)})$, $h, k = q_i - q_{i-1}, \dots, 1$ be a $q_i - q_{i-1}$ matrix over K such that the vectors

$$w_{\mu_{q_i},j}^1 = \sum_{k=q_i-q_{i-1}}^1 c_{j,k}^{(i)} v_{\mu_{q_i},k}^1$$

form a basis with respect to which $L_{X,i,1}$ is upper triangular. If we set

$$w_{\mu_{q_i},j}^l = \sum_{k=q_i-q_{i-1}}^1 c_{j,k}^{(i)} v_{\mu_{q_i},k}^l$$

for $l = 1, \dots, \mu_{q_i}$ and for $i = 1, \dots, u$ we get the basis required by ii). \square

We can shortly say that $X \in \mathcal{C}_B$ if and only if its blocks are upper triangular Toeplitz matrices.

By lemma 3.2 if $A \in \mathcal{C}_B$ then $A(i,l) = A(i,l')$ for $i \in \{1, \dots, u\}$ and $l, l' \in \{1, \dots, \mu_{q_i}\}$; we denote this matrix by $A(i)$.

We will denote by $\overline{\mathcal{C}}_B$ the subspace of all $A \in \mathcal{C}_B$ such that $A(i)$ is lower triangular for $i = 1, \dots, u$. Moreover we will set

$$\overline{\mathcal{N}}_B = \overline{\mathcal{C}}_B \cap \mathcal{N}_B.$$

Example If $B = (3, 3, 3, 2)$ we have that $A \in \mathcal{N}_B$ if and only if there exist

$$\begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{13}^1 \\ a_{21}^1 & a_{22}^1 & a_{23}^1 \\ a_{31}^1 & a_{32}^1 & a_{33}^1 \end{pmatrix} \in N(n, 3)$$

and $a_{h,k}^l \in K$, for $(h, k, l) \in \{1, 2, 3\} \times \{1, 2, 3\} \times \{2, 3\}$, for $(h, k, l) \in (\{4\} \times \{1, 2, 3\} \cup \{1, 2, 3\} \times \{4\}) \times \{1, 2\}$ and for $(h, k, l) = (4, 4, 2)$, such that A is the matrix:

$$\left(\begin{array}{ccc|ccc|ccc|cc} a_{11}^1 & a_{11}^2 & a_{11}^3 & & a_{12}^1 & a_{12}^2 & a_{12}^3 & & a_{13}^1 & a_{13}^2 & a_{13}^3 & & a_{14}^1 & a_{14}^2 \\ & a_{11}^1 & a_{11}^2 & & & a_{12}^1 & a_{12}^2 & & & a_{13}^1 & a_{13}^2 & & & a_{14}^1 \\ & & a_{11}^1 & & & & a_{12}^1 & & & & a_{13}^1 & & & \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ a_{21}^1 & a_{21}^2 & a_{21}^3 & & a_{22}^1 & a_{22}^2 & a_{22}^3 & & a_{23}^1 & a_{23}^2 & a_{23}^3 & & a_{24}^1 & a_{24}^2 \\ & a_{21}^1 & a_{21}^2 & & & a_{22}^1 & a_{22}^2 & & & a_{23}^1 & a_{23}^2 & & & a_{24}^1 \\ & & a_{21}^1 & & & & a_{22}^1 & & & & a_{23}^1 & & & \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ a_{31}^1 & a_{31}^2 & a_{31}^3 & & a_{32}^1 & a_{32}^2 & a_{32}^3 & & a_{33}^1 & a_{33}^2 & a_{33}^3 & & a_{34}^1 & a_{34}^2 \\ & a_{31}^1 & a_{31}^2 & & & a_{32}^1 & a_{32}^2 & & & a_{33}^1 & a_{33}^2 & & & a_{34}^1 \\ & & a_{31}^1 & & & & a_{32}^1 & & & & a_{33}^1 & & & \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ & a_{41}^1 & a_{41}^2 & & & a_{42}^1 & a_{42}^2 & & & a_{43}^1 & a_{43}^2 & & 0 & a_{44}^2 \\ & & a_{41}^1 & & & & a_{42}^1 & & & & a_{43}^1 & & & 0 \end{array} \right)$$

For this $A \in \mathcal{N}_B$ it is possible to choose $G \in \mathcal{C}_B$ and $\bar{a}_{h,k}^l \in K$, for $(h,k,l) \in \{1,2,3\}^3$ and $h > k$, for $(h,k,l) \in \{1,2,3\} \times \{1,2,3\} \times \{2,3\}$ and $h \leq k$, for $(h,k,l) \in (\{4\} \times \{1,2,3\} \cup \{1,2,3\} \times \{4\}) \times \{1,2\}$, such that $\det G \neq 0$ and $G^{-1}AG$ is the following element of $\bar{\mathcal{N}}_B$:

$$\left(\begin{array}{ccc|ccc|ccc|cc} 0 & \bar{a}_{11}^2 & \bar{a}_{11}^3 & & 0 & \bar{a}_{12}^2 & \bar{a}_{12}^3 & & 0 & \bar{a}_{13}^2 & \bar{a}_{13}^3 & & \bar{a}_{14}^1 & \bar{a}_{14}^2 \\ & 0 & \bar{a}_{11}^2 & & & 0 & \bar{a}_{12}^2 & & & 0 & \bar{a}_{13}^2 & & & \bar{a}_{14}^1 \\ & & 0 & & & & 0 & & & & 0 & & & \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ \bar{a}_{21}^1 & \bar{a}_{21}^2 & \bar{a}_{21}^3 & & 0 & \bar{a}_{22}^2 & \bar{a}_{22}^3 & & 0 & \bar{a}_{23}^2 & \bar{a}_{23}^3 & & \bar{a}_{24}^1 & \bar{a}_{24}^2 \\ & \bar{a}_{21}^1 & \bar{a}_{21}^2 & & & 0 & \bar{a}_{22}^2 & & & 0 & \bar{a}_{23}^2 & & & \bar{a}_{24}^1 \\ & & \bar{a}_{21}^1 & & & & 0 & & & & 0 & & & \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ \bar{a}_{31}^1 & \bar{a}_{31}^2 & \bar{a}_{31}^3 & & \bar{a}_{32}^1 & \bar{a}_{32}^2 & \bar{a}_{32}^3 & & 0 & \bar{a}_{33}^2 & \bar{a}_{33}^3 & & \bar{a}_{34}^1 & \bar{a}_{34}^2 \\ & \bar{a}_{31}^1 & \bar{a}_{31}^2 & & & \bar{a}_{32}^1 & \bar{a}_{32}^2 & & & 0 & \bar{a}_{33}^2 & & & \bar{a}_{34}^1 \\ & & \bar{a}_{31}^1 & & & & \bar{a}_{32}^1 & & & & 0 & & & \\ - & - & - & - & - & - & - & - & - & - & - & - & - & - \\ & \bar{a}_{41}^1 & \bar{a}_{41}^2 & & & \bar{a}_{42}^1 & \bar{a}_{42}^2 & & & \bar{a}_{43}^1 & \bar{a}_{43}^2 & & 0 & \bar{a}_{44}^2 \\ & & \bar{a}_{41}^1 & & & & \bar{a}_{42}^1 & & & & \bar{a}_{43}^1 & & & 0 \end{array} \right)$$

By lemma 3.1 we get the following result.

Corollary 3.1 *The subvariety $\bar{\mathcal{E}}_B(\bar{\mathcal{N}}_B)$ has nonempty intersection with the orbit of any element of $\mathcal{E}_B(\mathcal{N}_B)$.*

4 Upper triangular form and properties of $\overline{\mathcal{E}}_B, \overline{\mathcal{N}}_B$

We will denote by $<$ the order of Δ_B ; then we have that $v_{\mu_{q_i},j}^l < v_{\mu_{q_{i'}},j'}^{l'}$ iff one of the following conditions holds:

- c₁) $i < i'$ ($\mu_{q_i} > \mu_{q_{i'}}$);
- c₂) $i = i'$ and $j > j'$;
- c₃) $i = i'$, $j = j'$ and $l > l'$.

The block form $X = (X_{h,k})$, $h, k = 1, \dots, t$ for the elements of $M(n, K)$ corresponds to the map from Δ_B onto $\{1, \dots, t\}$ defined by $v_{\mu_{q_i},j}^l \mapsto q_i - j + 1$. By lemma 3.1 we get the following result.

Lemma 4.1 *For $i, i' \in \{1, \dots, u\}$, $j \in \{q_i - q_{i-1}, \dots, 1\}$ and $j' \in \{q_{i'} - q_{i'-1}, \dots, 1\}$ the maximum rank of $X_{q_i-j+1, q_{i'}-j'+1}$ for $X \in \overline{\mathcal{E}}_B$ ($X \in \overline{\mathcal{N}}_B$) is:*

- a) μ'_{q_i} if $i < i'$ ($\mu_{q_i} > \mu_{q_{i'}}$);
- b) $\mu_{q_i} - 1$ if $i = i'$ and $j \leq j'$;
- c) μ_{q_i} if $i > i'$ and if $i = i'$, $j > j'$.

If X is an endomorphism of K^n and Λ is basis of K^n we will denote by $\mathcal{R}_{X,\Lambda}$ the relation in the set of the elements of Λ defined as follows: $w' \mathcal{R}_{X,\Lambda} w$ iff $X w'$ has nonzero entry with respect to w .

Corollary 4.1 *There exists a nonempty open subset of $\overline{\mathcal{E}}_B$ ($\overline{\mathcal{N}}_B$) such that if X belongs to it and $v_{\mu_{q_i},j}^l, v_{\mu_{q_{i'}},j'}^{l'} \in \Delta_B$ then $v_{\mu_{q_{i'}},j'}^{l'} \mathcal{R}_{X,\Delta_B} v_{\mu_{q_i},j}^l$ iff one of the following conditions holds:*

- ι_1) $i < i'$ and $\mu_{q_i} - l \leq \mu_{q_{i'}} - l'$;
- ι_2) $i = i'$, $j \geq j'$ and $\mu_{q_i} - l < \mu_{q_{i'}} - l'$ (that is $l > l'$);
- ι_3) $i > i'$ and $l \geq l'$, or $i = i'$, $j < j'$ and $l \geq l'$.

The subalgebra $\overline{\mathcal{E}}_B$ ($\overline{\mathcal{N}}_B$) is a maximal nilpotent subalgebra of \mathcal{D}_B (\mathcal{C}_B); hence there exists a basis of K^n with respect to which all the elements of $\overline{\mathcal{E}}_B$ ($\overline{\mathcal{N}}_B$) are upper triangular. Corollary 4.1 suggests that, in order to get such a basis, it is enough to change the order of Δ_B .

Let $v_{\mu_{q_i},j}^l \prec v_{\mu_{q_{i'}},j'}^{l'}$ iff one of the following conditions holds:

- $e_1)$ $\mu_{q_i} - l < \mu_{q_{i'}} - l'$;
- $e_2)$ $\mu_{q_i} - l = \mu_{q_{i'}} - l'$ and $i < i'$ (that is $l > l'$);
- $e_3)$ $\mu_{q_i} - l = \mu_{q_{i'}} - l'$, $i = i'$ (that is $l = l'$) and $j > j'$.

Let $\Delta_{B, \prec}$ be the basis of K^n which has the same elements as Δ_B with the order \prec . Then by corollary 4.1 we have that the representation of all the elements of $\overline{\mathcal{E}}_B$ ($\overline{\mathcal{N}}_B$) with respect to $\Delta_{B, \prec}$ is upper triangular.

We could define another similar order in Δ_B , which is the following: let $v_{\mu_{q_i}, j}^l \ll v_{\mu_{q_{i'}}, j'}^{l'}$ iff one of the following conditions holds:

- $d_1)$ $l > l'$,
- $d_2)$ $l = l'$ and $i > i'$ (that is $\mu_{q_i} - l < \mu_{q_{i'}} - l'$),
- $d_3)$ $l = l'$, $i = i'$ (that is $\mu_{q_i} - l = \mu_{q_{i'}} - l'$) and $j < j'$.

We will denote by $\Delta_{B, \ll}$ the basis of K^n which has the same elements as Δ_B with the order \ll .

Example Let $n = 13$ and $B = (4, 3^2, 2, 1)$. Let us consider the endomorphism \overline{A} of K^{13} which is represented, with respect to the basis

$$\Delta_B = \{\underbrace{v_{4,1}^4, v_{4,1}^3, v_{4,1}^2, v_{4,1}^1}_{\text{group 1}}, \underbrace{v_{3,2}^3, v_{3,2}^2, v_{3,2}^1}_{\text{group 2}}, \underbrace{v_{3,1}^3, v_{3,1}^2, v_{3,1}^1}_{\text{group 3}}, \underbrace{v_{2,1}^2, v_{2,1}^1}_{\text{group 4}}, \underbrace{v_{1,1}^1}_{\text{group 5}}\} ,$$

by the following matrix:

$$\left(\begin{array}{cccc|cccc|cccc|cc|c} 0 & a & b & c & p & q & r & v & z & w & \alpha & \beta & \lambda \\ & 0 & a & b & & p & q & & v & z & & \alpha & 0 \\ & & 0 & a & & & p & & & v & & & \\ & & & 0 & & & & & & & & & \\ - & - & - & - & - & - & - & - & - & - & - & - & - \\ & s & t & u & 0 & i & l & 0 & m & n & \rho & \sigma & \pi \\ & & s & t & & 0 & i & & 0 & m & & \rho & 0 \\ & & & s & & & 0 & & & 0 & & & \\ - & - & - & - & - & - & - & - & - & - & - & - & - \\ & j & y & k & d & e & f & 0 & g & h & \xi & \zeta & \theta \\ & & j & y & & d & e & & 0 & g & & \xi & 0 \\ & & & j & & & d & & & 0 & & & \\ - & - & - & - & - & - & - & - & - & - & - & - & - \\ & \mathbf{0} & \delta & \epsilon & & \eta & \nu & & \tau & \gamma & 0 & o & \phi \\ & & \mathbf{0} & \delta & & & \eta & & & \tau & & 0 & \\ - & - & - & - & - & - & - & - & - & - & - & - & - \\ & \mathbf{0} & \mathbf{0} & \omega & & \mathbf{0} & \psi & & \mathbf{0} & \iota & & \chi & 0 \end{array} \right) .$$

In this case we have

$$\Delta_{B,\prec} = \{\underbrace{v_{4,1}^4, v_{3,1}^3, v_{3,2}^3, v_{2,1}^2, v_{1,1}^1}_{\text{set 1}}, \underbrace{v_{4,1}^3, v_{3,1}^2, v_{3,2}^2, v_{2,1}^1}_{\text{set 2}}, \underbrace{v_{4,1}^2, v_{3,1}^1, v_{3,2}^1}_{\text{set 3}}, \underbrace{v_{4,1}^1}_{\text{set 4}}\} ;$$

$$\Delta_{B,\ll} = \{\underbrace{v_{4,1}^4, v_{3,1}^3, v_{3,2}^3, v_{4,1}^3}_{\text{set 1}}, \underbrace{v_{2,1}^2, v_{3,1}^2, v_{3,2}^2, v_{4,1}^2}_{\text{set 2}}, \underbrace{v_{1,1}^1, v_{2,1}^1, v_{3,1}^1, v_{3,2}^1, v_{4,1}^1}_{\text{set 3}}\} .$$

The matrix of \overline{A} with respect to $\Delta_{B,\prec}$ is the following:

$$\begin{pmatrix} 0 & v & p & \alpha & \lambda & | & a & z & q & \beta & | & b & w & r & | & c \\ & 0 & d & \xi & \theta & | & j & g & e & \zeta & | & y & h & f & | & k \\ & & 0 & \rho & \pi & | & s & m & i & \sigma & | & t & n & l & | & u \\ & & & 0 & \phi & | & \mathbf{0} & \tau & \eta & o & | & \delta & \gamma & \nu & | & \epsilon \\ & & & & 0 & | & \mathbf{0} & \mathbf{0} & \mathbf{0} & \chi & | & \mathbf{0} & \iota & \psi & | & \omega \\ - & - & - & - & - & | & - & - & - & - & | & - & - & - & | & - \\ & & & & & | & 0 & v & p & \alpha & | & a & z & q & | & b \\ & & & & & | & & 0 & d & \xi & | & j & g & e & | & y \\ & & & & & | & & & 0 & \rho & | & s & m & i & | & t \\ & & & & & | & & & & 0 & | & \mathbf{0} & \tau & \eta & | & \delta \\ - & - & - & - & - & | & - & - & - & - & | & - & - & - & | & - \\ & & & & & | & & & & & | & 0 & v & p & | & a \\ & & & & & | & & & & & | & & 0 & d & | & j \\ & & & & & | & & & & & | & & & 0 & | & s \\ - & - & - & - & - & | & - & - & - & - & | & - & - & - & | & - \\ & & & & & | & & & & & | & & & & | & 0 \end{pmatrix} ;$$

the matrix of \overline{A} with respect to $\Delta_{B,\ll}$ is the following:

$$\begin{pmatrix} 0 & | & v & p & a & | & \alpha & z & q & b & | & \lambda & \beta & w & r & c \\ - & | & 0 & d & j & | & \xi & g & e & y & | & \theta & \zeta & h & f & k \\ & | & & 0 & s & | & \rho & m & i & t & | & \pi & \sigma & n & l & u \\ & | & & & 0 & | & \mathbf{0} & v & p & a & | & \mathbf{0} & \alpha & z & q & b \\ - & | & - & - & - & | & 0 & \tau & \eta & \delta & | & \phi & o & \gamma & \nu & \epsilon \\ & | & & & & | & & 0 & d & j & | & \mathbf{0} & \xi & g & e & y \\ & | & & & & | & & & 0 & s & | & \mathbf{0} & \rho & m & i & t \\ & | & & & & | & & & & 0 & | & \mathbf{0} & \mathbf{0} & v & p & a \\ - & | & - & - & - & | & - & - & - & - & | & 0 & \chi & \iota & \psi & \omega \\ & | & & & & | & & & & & | & & 0 & \tau & \eta & \delta \\ & | & & & & | & & & & & | & & & 0 & d & j \\ & | & & & & | & & & & & | & & & & 0 & s \\ & | & & & & | & & & & & | & & & & & 0 \end{pmatrix} .$$

In this section we will represent any endomorphism of K^n with respect to the basis $\Delta_{B, \prec}$. For $h = 0, \dots, \mu_{q_1} - 1$ let

$$\Delta_{B, \prec, h} = \{v_{\mu_{q_i}, j}^l \in \Delta_B \mid \mu_{q_i} - l = h\},$$

with the order induced by \prec . We have that

$$|\Delta_{B, \prec, h}| > |\Delta_{B, \prec, h'}| \quad \text{if} \quad h < h'.$$

Let π_h be the canonical projection of K^n onto $\langle \Delta_{B, \prec, h} \rangle$; for $X \in M(n, K)$ and $h, k \in \{0, \dots, \mu_{q_1} - 1\}$ let $X_{h,k} = \pi_h \circ X|_{\langle \Delta_{B, \prec, k} \rangle}$; we consider X as a block matrix:

$$X = (X_{h,k}), \quad h, k \in \{0, \dots, \mu_{q_1} - 1\}.$$

Let $A \in \overline{\mathcal{N}}_B$. If we cancel the column of the entries of $v_{\mu_{q_1}, 1}^l$ and the row of the entries with respect to $v_{\mu_{q_1}, 1}^l$ for $l = 1, \dots, \mu_{q_1}$ (that is the first row of any row of blocks and the first column of any column of blocks) we get a matrix A' of $\overline{\mathcal{N}}_{B'}$, where $B' = (\mu_2, \dots, \mu_t)$. If we instead cancel the column of the entries of $v_{\mu_{q_i}, j}^1$ and the row of the entries with respect to $v_{\mu_{q_i}, j}^1$ for $i = 1, \dots, u$ and $j = 1, \dots, q_i - q_{i-1}$ we get a matrix \tilde{A} of $\overline{\mathcal{N}}_{\tilde{B}}$, where $\tilde{B} = (\mu_1 - 1, \dots, \mu_t - 1)$ (here we cancel the 0's).

Proposition 4.1 *$\overline{\mathcal{N}}_B$ is the affine space of all the upper triangular matrices $A \in N(n, K)$ such that:*

- i) *the entry of $Av_{\mu_{q_{i'}}, j'}^{l'}$ with respect to $v_{\mu_{q_i}, j}^l$ is 0 if $\mu_{q_{i'}} - \mu_{q_i} > 1$ and $\mu_{q_{i'}} - \mu_{q_i} > l' - l$;*
- ii) *for $k \in \{1, \dots, \mu_{q_1} - 1\}$ and $h \in \{1, \dots, k\}$ the entry of $A_{h,k}$ of indices (i, j) is equal to the entry of $A_{h-1, k-1}$ of indices (i, j) .*

Proof The claim can be proved by lemma 3.2, corollary 4.1 and induction on n (we can consider A' or \tilde{A}). \square

Corollary 4.2 *For $h, k \in \{0, \dots, \mu_{q_1} - 1\}$ and $m \in \mathbb{N}$ we have that:*

- iii) *if $Y \in M(|\Delta_{B, \prec, h}|, K)$ is strictly upper triangular there exists $A \in \overline{\mathcal{N}}_B$ such that $(A^m)_{h,h} = Y^m$;*
- iv) *if the entry of $(A^m)_{h,k}$ of indices (i, j) is 0 for all $A \in \overline{\mathcal{N}}_B$ then, for all $A \in \overline{\mathcal{N}}_B$, the entry of $(A^m)_{h,k}$ of indices (i', j') is 0 for $i' = i, \dots, |\Delta_{B, \prec, h}|$ and $j' = 1, \dots, j$; moreover the entry of indices (i, j) of $(A^m)_{h, k-1}$ (if $k \neq 0$) and of $(A^m)_{h+1, k}$ (if $h \neq \mu_{q_1} - 1$ and $i \leq |\Delta_{B, \prec, h+1}|$) is also 0.*

Proof We can prove the claim by induction on n and corollary 4.1. By ii) of proposition 4.1, for the second claim of iv) it is enough to prove that if $k \in \{1, \dots, \mu_{q_1} - 1\}$ and the entry of indices (i, j) of $(A^m)_{0,k}$ is 0 for all $A \in \overline{\mathcal{N}}_B$ then the entry of indices (i, j) of $(A^m)_{0,k-1}$ is also 0 for all $A \in \overline{\mathcal{N}}_B$. We can prove this by induction on $|\Delta_{B, \prec, 0}| - i$ (if $i = |\Delta_{B, \prec, 0}|$ the claim is true, since $(A')^{m-1}$ has the property iv) by the inductive hypothesis). \square

We set $\{e_1, \dots, e_n\} = \Delta_{B, \prec}$, that is the basis with respect to which we now represent the elements of $\overline{\mathcal{E}}_B$. For $i \in \{1, \dots, n\}$ let

$$I_i = \{l \in \{1, \dots, n\} \mid x_{i,l} \text{ isn't an identity on } \overline{\mathcal{E}}_B\}.$$

Let Φ_B be the map in $\{1, \dots, n+1\}$ defined as follows:

$$\Phi_B = \begin{cases} \min I_i & \text{if } i \in \{1, \dots, n\} \text{ and } I_i \neq \emptyset \\ n+1 & \text{if } i \in \{1, \dots, n\} \text{ and } I_i = \emptyset, \text{ or } i = n+1. \end{cases}$$

We have that $\overline{\mathcal{E}}_B$ is a subvariety of N_{Φ_B} . We will denote by $\widetilde{\overline{\mathcal{E}}}_B$ the open subset of $\overline{\mathcal{E}}_B$ described in corollary 2.1, which is contained in the maximum nilpotent orbit which has elements in $\overline{\mathcal{E}}_B$.

Proposition 4.2 *The open subset $\widetilde{\overline{\mathcal{E}}}_B$ of $\overline{\mathcal{E}}_B$ has nonempty intersection with $\overline{\mathcal{N}}_B$.*

Proof If B is almost rectangular there exist in $\overline{\mathcal{N}}_B$ elements whose rank is $n-1$, hence we can prove the claim by induction on n . For any $X = (x_{i,j}) \in \overline{\mathcal{E}}_B$ let \tilde{X} be the submatrix obtained by choosing the last $n-t$ rows and columns; if we set $\tilde{B} = (\mu_1 - 1, \dots, \mu_t - 1)$ (omitting the 0's) then the space of the matrices \tilde{X} is $\overline{\mathcal{E}}_{\tilde{B}}$. We assume that $X \in \overline{\mathcal{N}}_B$ and, by the inductive hypothesis, that \tilde{X} satisfies the conditions which define the open subset $\widetilde{\overline{\mathcal{E}}}_{\tilde{B}}$ of $\overline{\mathcal{E}}_{\tilde{B}}$. For $h = 0, \dots, \mu_{q_1} - 1$ let

$$t_h = \sum_{l=0}^h |\Delta_{B, \prec, h}|;$$

then $x_{i,i+1} \neq 0$ for all $i \in \{1, \dots, n\} - \{t_h, h = 0, \dots, \mu_{q_1} - 1\}$. To $\widetilde{\overline{\mathcal{E}}}_{\tilde{B}}$ it corresponds $\tilde{\phi}$ which satisfy the claim of corollary 2.1. We change the basis $\{e_1, \dots, e_n\}$ in the following way, still denoting by X the representation of the matrix with respect to the new basis: for $i = n-1, \dots, t_1 + 1$

we add to e_i a linear combination of the vectors e_{t_h} such that $t_h < i$, in such a way that $\tilde{X} \in \mathcal{A}_{\tilde{\phi}}$, as it is claimed in a) of lemma 2.3. Let $\phi \in I_B$ be the map which corresponds to $\bar{\mathcal{E}}_B$ according to corollary 2.1; if $\phi(t_0) = n + 1$ the claim is obvious, hence we assume that $\phi(t_0) \neq n + 1$. Then there exists $\bar{h} \in \{0, \dots, \mu_{q_1} - 1\}$ such that $\phi(t_0) = t_{\bar{h}} + 1$. For $h \in \{0, \dots, \mu_{q_1} - 1\}$ let C_h be the submatrix of X obtained by choosing the rows of indices $t_0, t_0 + 1, \dots, t_h$ and the columns of indices $t_0 + 1, t_0 + 2, \dots, t_h + 1$. The condition $\phi(t_0) = t_{\bar{h}} + 1$ is equivalent to the following condition: for all $X \in \bar{\mathcal{E}}_B$ the first row of C_h is a linear combination of the other rows of C_h for $h = 0, \dots, \bar{h} - 1$, while there exists $X \in \bar{\mathcal{E}}_B$ such that the first row of $C_{\bar{h}}$ isn't a linear combination of the other rows of $C_{\bar{h}}$. But there exists $X \in \bar{\mathcal{N}}_B$ and $l \in \{t_0 + 1, \dots, t_1 - 1\}$ such that $x_{l, t_{\bar{h}} + 1}$ doesn't satisfy with other entries of $C_{\bar{h}}$ any of the equations of $\bar{\mathcal{N}}_B$ as subvariety of $\bar{\mathcal{E}}_B$, hence there exists $X \in \bar{\mathcal{N}}_B$ such that the first row of $C_{\bar{h}}$ isn't a linear combination of the other rows of $C_{\bar{h}}$ (that is, the Toeplitz conditions of lemma 3.2 don't imply that linear dependence). \square

By propositions 4.2 we get the following result.

Corollary 4.3 *There exists a nonempty open subset of $\bar{\mathcal{N}}_B \times \bar{\mathcal{E}}_B$ such that if (A, X) belongs to it we have*

$$\text{rank } A^m = \text{rank } X^m \quad \text{for all } m \in \mathbb{N}.$$

Proof The claim is true for the subset $(\widetilde{\bar{\mathcal{E}}_B} \cap \bar{\mathcal{N}}_B) \times \widetilde{\bar{\mathcal{E}}_B}$. \square

5 The graph associated to B

Let \mathcal{R}_B be the relation in the set of the elements of Δ_B defined as follows:

$$v_{\mu_{q_{i'}}}^{l'} \mathcal{R}_B v_{\mu_{q_i}}^l \iff \iota_1 \text{ or } \iota_2 \text{ or } \iota_3 \text{ of corollary 4.1 holds.}$$

Proposition 5.1 *The relation \mathcal{R}_B in the set of the elements of Δ_B is a strict partial order.*

Proof The relation \mathcal{R}_B is obviously antisymmetric; the condition ι_1) implies $l > l'$, hence it is also transitive. \square

The following result is a consequence of corollary 4.3.

Corollary 5.1 *The maximum nilpotent orbit of the elements of $\bar{\mathcal{N}}_B$ is determined by the relation \mathcal{R}_B .*

We will write the vertices of the graph of \mathcal{R}_B in such a way that they form a table as follows. The indices of the rows are the elements of $\mathbb{N} \cup \{0\}$ and the columns have as indices $1, \dots, u$; the entries are the elements of Δ_B : $v_{\mu_{q_i}, j}^l$ is written in the i -th column and in the row whose index is the maximum number h such that there exist elements of Δ_B whose images under X^h have nonzero entry with respect to $v_{\mu_{q_i}, j}^l$ for some $X \in \overline{\mathcal{E}}_B$. The graph of \mathcal{R}_B can be obtained by writing arrows on this table according to corollary 4.1. We will say that this table is the "graph of B ". We show it in the following examples.

Examples The "graph of B " for $B = (3^2, 2, 1)$ and for $B = (6^4, 3^2, 2, 1)$ is respectively as follows:

	1	2	3	6
0				v_{64}^1
1				v_{63}^1
2				v_{62}^1
3				v_{61}^1
4			v_{32}^1	v_{64}^2
5			v_{31}^1	v_{63}^2
6		v_{21}^1		v_{62}^2
7	v_{11}^1			v_{61}^2
8			v_{32}^2	v_{64}^3
9			v_{31}^2	v_{63}^3
10		v_{21}^2		v_{62}^3
11				v_{61}^3
12			v_{32}^3	v_{64}^4
13			v_{31}^3	v_{63}^4
14				v_{62}^4
15				v_{61}^4
16				v_{64}^5
17				v_{63}^5
18				v_{62}^5
19				v_{61}^5
20				v_{64}^6
21				v_{63}^6
22				v_{62}^6
23				v_{61}^6

For $B = (7, 4, 2)$, $B = (2^2, 1)$ and $B = (4, 2^2, 1)$ it is respectively as follows:

	2	4	7		1	2		1	2	4
0			v_{71}^1					0		v_{41}^1
1		v_{41}^1	v_{71}^2		0	v_{22}^1		1	v_{22}^1	v_{41}^2
2	v_{21}^1	v_{41}^2	v_{71}^3		1	v_{21}^1		2	v_{21}^1	
3	v_{21}^2	v_{41}^3	v_{71}^4		2	v_{11}^2		3	v_{11}^1	v_{41}^3
4		v_{41}^4	v_{71}^5		3	v_{22}^2		4	v_{22}^2	
5		v_{41}^5	v_{71}^6		4	v_{21}^2		5	v_{21}^2	
6			v_{71}^7					6		v_{41}^4

For $B = (4, 3^2, 2, 1)$ and $B = (5, 4, 3^2, 2, 1)$ it is respectively

	1	2	3	4		1	2	3	4	5
0				v_{41}^1		0				v_{51}^1
1			v_{32}^1			1			v_{41}^1	
2			v_{31}^1			2		v_{32}^1		v_{51}^2
3		v_{21}^1		v_{41}^2		3		v_{31}^1		
4	v_{11}^1		v_{32}^2			4	v_{21}^1	v_{32}^2	v_{41}^2	
5			v_{31}^2			5	v_{11}^1	v_{32}^2		v_{51}^3
6		v_{21}^2		v_{41}^3		6		v_{31}^2		
7			v_{32}^3			7	v_{21}^2	v_{32}^3	v_{41}^3	
8			v_{31}^3			8		v_{31}^3		v_{51}^4
9				v_{41}^4		9			v_{41}^4	
						10				v_{51}^5
						11				

For $B = (2^3)$, for $B = (5, 2^3)$ and for $B = (6, 5, 2^3)$ it is respectively

	2		2	5		2	5	6
			0	v_{51}^1		0		v_{61}^1
0	v_{23}^1		1	v_{23}^1		1	v_{51}^1	
1	v_{22}^1		2	v_{22}^1		2	v_{23}^1	v_{61}^2
2	v_{21}^1		3	v_{21}^1		3	v_{22}^1	v_{51}^2
3	v_{23}^2		4	v_{23}^2		4	v_{21}^1	v_{61}^3
4	v_{22}^2		5	v_{22}^2	v_{51}^4	5	v_{23}^2	v_{51}^3
5	v_{21}^2		6	v_{21}^2		6	v_{22}^2	v_{61}^4
			7	v_{51}^5		7	v_{21}^2	v_{51}^4
						8		v_{61}^5
						9	v_{51}^5	
						10		v_{61}^6

Let $\nu_1 \in \mathbb{N}$ be such that $\nu_1 - 1$ is the index of the last row of the "graph of B " where there are written some vectors.

Let \mathcal{F}_B be the set of all the endomorphisms X' of K^n such that the graph of $\mathcal{R}_{X', \Delta_B}$ is obtained by adding to the "graph of B " all the arrows which link any element of the m -th row with any element of the m' -th row for $m, m' \in \{0, \dots, \nu_1 - 1\}$ such that $m < m'$.

Proposition 5.2 *For all $X' \in \mathcal{F}_B$ there exists an element of $\overline{\mathcal{E}}_B$ which is conjugated to X' .*

Proof For $m = 1, \dots, \nu_1 - 1$ let R_m be the subspace of K^n generated by the elements of Δ_B which are written in the rows of the "graph of B " of indices greater or equal than m . We can replace any vector of $\Delta_B \cap R_m$ with the sum of it and another element of R_m , getting a new basis of K^n . By induction on $\nu_1 - m + 1$ the endomorphism $X \in \overline{\mathcal{E}}_B$ and this element can be chosen such that the representation of X with respect to this new basis is the matrix of X' with respect to Δ_B . \square

Let X be the matrix of an endomorphism of K^n with respect to the basis Δ_B ; we will consider pairs $(X, v) \in N(n, K) \times K^n$ with the following property:

$C_1) : \forall m, i \in \mathbb{N} \cup \{0\}$ such that $m \leq i \leq \nu_1 - 1$ the entry of $X^m v$ with respect to any vector of the i -th row of the "graph of B " isn't 0.

Let \mathcal{K}_B be the subset of K^n of all the vectors which have nonzero entry with respect to $v_{\mu_{q_1}, q_1}^1$.

Proposition 5.3 *The projection on K^n of the subset of $\overline{\mathcal{E}}_B \times K^n$ ($\mathcal{F}_B \times K^n$) of all the pairs with property C_1) is \mathcal{K}_B .*

Proof By corollary 4.1 there exists a nonempty open subset of $\overline{\mathcal{E}}_B$ (\mathcal{F}_B) such that if X belongs to it the vector $X v_{\mu_{q_1}, q_1}^1$ has nonzero entry with respect to all the elements of $\Delta_B - \{v_{\mu_{q_1}, q_1}^1\}$. Hence if $v \in K^n$, $m, i \in \mathbb{N} \cup \{0\}$, v has nonzero entry with respect to $v_{\mu_{q_1}, q_1}^1$ and $i \geq m$ the condition that $X^m v$ has zero entry with respect to a vector of the i -th row of the "graph of B " isn't an identity in X . \square

6 The maximum partition in \mathcal{N}_B (\mathcal{E}_B)

We will consider pairs $(X, v) \in N(n, K) \times K^n$ with the following property:

$$C_2) : K^n = \langle X^h J^k v, h, k = 1, \dots, n \rangle$$

(that is v is cyclic for (X, J)). The following proposition explains a generalization of a known result for the elements of \mathcal{N}_B .

Proposition 6.1 *The projection on K^n of the subset of $\overline{\mathcal{E}}_B \times K^n$ ($\mathcal{F}_B \times K^n$) of all the pairs with the property C_2) contains \mathcal{K}_B .*

Proof In [11] it has been proved that the subset of $\overline{\mathcal{N}}_B \times K^n$ of all the pairs with the property C_2) is nonempty. The projection of this subset on K^n is a nonempty open subset, hence it has nonempty intersection with \mathcal{K}_B . Since any element of \mathcal{K}_B can be the μ_{q_1} -th element of a Jordan basis for J we get that \mathcal{K}_B is contained in that projection. \square

For $X \in \overline{\mathcal{E}}_B$ let \mathcal{K}_X be the subset of K^n of all the vectors which are cyclic for (X, J) .

Proposition 6.2 *Let $X \in \overline{\mathcal{E}}_B$ and let $v \in \mathcal{K}_X$.*

i) X has partition (ν_1, \dots, ν_z) iff the set

$$\Omega_{(X, J)} = \{ X^h J^k v \mid h, k \in \mathbb{N} \cup \{0\}, h \leq \nu_{k+1} - 1, k \leq z - 1 \}$$

is a Jordan basis for X ;

ii) if X has partition (ν_1, \dots, ν_z) , $h', k' \in \mathbb{N} \cup \{0\}$ and $k' > z - 1$ or $h' > \nu_{k'+1} - 1$ then

$$X^{h'} J^{k'} v \in \langle X^h J^k v \mid h, k \in \mathbb{N} \cup \{0\}, h \leq \nu_{k+1} - 1, k \leq k' - 1 \rangle.$$

Proof Let $W_0 = \{0\}$ and let $W_i = \langle X^h J^k v \mid h, k \in \mathbb{N} \cup \{0\}, k < i \rangle$ for $i = 1, \dots, z - 1$. W_i is stable with respect to X and

$$\frac{K^n}{W_i} = \langle W_i + X^h J^k v \mid h, k \in \mathbb{N} \cup \{0\}, k \geq i \rangle.$$

Let X_i be the endomorphism of $\frac{K^n}{W_i}$ defined by $X_i(W_i + w) = W_i + Xw$. Let us assume that X has partition (ν_1, \dots, ν_z) . Since ν_1 is the index of nilpotency of X we have $X^{\nu_1-1}v \neq 0$, hence $W_1 = \langle v, Xv, \dots, X^{\nu_1-1}v \rangle$. Then (by a classical proof of the existence of a Jordan basis) ν_2 is the index of nilpotency of X_1 . Hence $X^{\nu_2-1}Jv \notin W_1$. Similarly for $i = 2, \dots, z - 1$ we have that ν_{i+1} is the index of nilpotency of X_i and then $X^{\nu_{i+1}-1}J^i v \notin W_i$. \square

The following corollary explains a result proved by Polona Oblak in [12].

Corollary 6.1 *The map from Δ_B° to $\{0, \dots, \nu_1 - 1\}$ which associates to any vector the index of its row in the "graph of B " is a bijection.*

Proof It is a consequence of corollary 4.1 and proposition 5.3. \square

Let $(X, v) \in \mathcal{F}_B \times K^n$ be a pair with the properties $C_1)$ and $C_2)$; let:

$$W_1 = \langle v, Xv, \dots, X^{\nu_1-1}v \rangle .$$

Lemma 6.1 *We have that $W_1 \cap \langle \Delta_B - \Delta_B^\circ \rangle = \{0\}$.*

Proof Let $w \in W_1 - \{0\}$ and let h' be the minimum of the set

$$\{h \in \{0, \dots, \nu_1 - 1\} \mid w \text{ has nonzero entry with respect to } A^{h'}v\} .$$

By corollary 4.1 w has nonzero entry with respect to the element of Δ_B° which is written in the row of index h' ; hence $w \notin \langle \Delta_B - \Delta_B^\circ \rangle$. \square

Let

$$\widehat{\Delta}_B = \{v + W_1 \mid v \in \Delta_B - \Delta_B^\circ\}$$

and let X_1 be the endomorphism of $\frac{K^n}{W_1}$ defined by $\widehat{X}(w + W_1) = X(w) + W_1$;

we can consider the relation $\mathcal{R}_{X_1, \widehat{\Delta}_B}$ in the set of the elements of $\widehat{\Delta}_B$ which is associated to X_1 .

Theorem 6.1 *For all $X' \in \mathcal{F}_{\widehat{B}}$ there exists $X \in \mathcal{F}_B$ such that the graph obtained from the graph of $\mathcal{R}_{X_1, \widehat{\Delta}_B}$ by the following changes:*

$\gamma_1)$ replacing the vertex $v_{\mu_{q_i}, j}^l + W_1$ with $v_{\mu_{q_i}, j}^l$ for $i = \tilde{i} + s + 1, \dots, u$,
 $j = q_i - q_{i-1}, \dots, 1$, $l = 1, \dots, \mu_{q_i}$,

$\gamma_2)$ replacing the vertex $v_{\mu_{q_i}, j}^l + W_1$ with $v_{\mu_{q_i}-2, j}^{l-1}$ for $i = 1, \dots, \tilde{i} - 1$, $j =$
 $q_i - q_{i-1}, \dots, 1$, $l = 2, \dots, \mu_{q_i} - 1$

is the graph of $\mathcal{R}_{X', \Delta_{\widehat{B}}}$.

Proof For $i = 1, \dots, u$, $j = q_i - q_{i-1}, \dots, 1$, $l = 1, \dots, \mu_{q_i}$ let $h(i, j, l)$ be the index of the row of the "graph of B " where $v_{\mu_{q_i}, j}^l$ is written. Let us represent $v_{\mu_{q_i}, j}^l + W_1$ with respect to the basis $\widehat{\Delta}_B$. The following claim can be proved by induction on $\nu_1 - h(i, j, l)$: if $v_{\mu_{q_i}, j}^l \in \Delta_B^\circ$ and $v_{\mu_{q_i}, j}^l + W_1$ has nonzero entry with respect to $v_{\mu_{q_{i'}}, j'}^{l'} + W_1$ (where $i' \in \{1, \dots, \tilde{i} - 1, \tilde{i} + s + 1, \dots, u\}$,

$j' \in \{q_{i'} - q_{i'-1}, \dots, 1\}$, $l = 1, \dots, \mu_{i'}$) then $h(i', j', l) \geq h(i, j, l)$. In fact, the claim is obvious if $(i, j, l) = (1, 1, \mu_{q_1})$ (that is $h(i, j, l) = \nu_1 - 1$); moreover there exists $a_{(i,j,l)} \in K$ such that $v_{\mu_{q_i}, j}^l - a_{(i,j,l)} X^{h(i,j,l)} v$ is a linear combination of the vectors of Δ_B which are written in the rows of the "graph of B " of indices greater or equal than $h(i, j, l)$, which by the inductive hypothesis implies the claim. Hence the graph which we obtain by the changes γ_1) and γ_2) has all the arrows required for being the graph of $\mathcal{R}_{X', \Delta_{\hat{B}}}$ for some $X' \in \mathcal{F}_{\hat{B}}$; moreover in this way we can get the graph corresponding to any element of $X' \in \mathcal{F}_{\hat{B}}$. \square

Now we can prove theorem 1.5, which was announced in the first section.

Proof of Theorem 1.5 It is a consequence of corollaries 3.1, 5.1, proposition 5.2 and theorem 6.1. \square

Example If $B = (5, 4, 3^2, 2, 1)$ we have $\nu_1 = 12$ and $\hat{B} = (5 - 2, 2, 1) = (3, 2, 1)$. Since the maximum partition of the elements of $\mathcal{N}_{\hat{B}}$ is $(5, 1)$ we get that the maximum partition of the elements of \mathcal{N}_B is $(12, 5, 1)$.

Theorem 1.5 leads to an algorithm for the determination of the maximum partition which is associated to elements of \mathcal{N}_B for any partition B .

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